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High-order residual distribution schemes on quadrilateral meshes

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SUMMARY

This article deals with the latest development in the construction of monotone high-order residual distribution schemes (RDS) on quadrilateral meshes. In the first part we give some generalities about the way upwind high-order RDS is designed. In the second part, some standard schemes are described. Then, the design of a stabilization to help the convergence is shown. The last part of this article is dedicated to the results obtained with these schemes. Copyright © 2007 John Wiley & Sons, Ltd.

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1. GENERALITIES AND NOTATIONS

We present the extension of residual distribution schemes (RDS) on high-order quadrilateral meshes to approximate steady solutions of

$$\frac{\partial u}{\partial t} + \nabla \cdot \mathscr{F}(u) = 0 \quad \forall (x, y) \in \Omega \tag{1}$$

on τ_h, an unstructured quadrilateral mesh of Ω. We use the standard Lagrangian Q^k elements such that the solution is approximated by a combination of the Lagrangian functions over the mesh. These Q^k elements are constructed such that in each quadrilateral E there are $(k+1)^2$ sub-elements, denoted by E_s and k^2 degrees of freedom σ_ℓ. We denote by ψ_i the continuous Lagrangian basis function at the degree of freedom σ_i; the solution u being approximated by

$$u^h(x, y) = \sum_{i \in \tau_h} \psi_i(x, y) u_i \tag{2}$$

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where u_i is the value of u at the degree of freedom σ_i. We consider the following iterative scheme:

$$u_i^{n+1} = u_i^n - \delta_i \sum_{E_s, i \in E_s} \Phi_i^{E_s}$$

where δ_i is an iterative parameter. We consider schemes where the distributed residuals verify

$$\sum_{j \in E_s} \Phi_j^{E_s} = \int_{E_s} \nabla \cdot \mathscr{F}(u^h)\, dx\, dy \tag{3}$$

We denote by $\beta_i^{E_s}$ the distribution coefficient defined by $\beta_i^{E_s} = \Phi^{E_s}/\Phi_i^{E_s}$. The schemes that we consider here verify the following properties.

1.1. Properties

Accuracy: To have a $(k+1)$-th order scheme, it is necessary to have the following truncation error:

$$\Phi_j^{E_s} = \mathcal{O}(h^{k+2})$$

for a smooth solution of (1).

Monotonicity: Rigorously, the definition of monotonic $\mathscr{RD}$s is linked to the theory of positive coefficients. In this paper we consider some schemes that behave as monotone schemes in the sense that they are quasi non-oscillatory schemes, but they are not rigorously monotone.

Upwinding: When a scheme on a triangulation is considered, the upwind parameter is defined by

$$k_i = \tfrac{1}{2} \boldsymbol{\lambda}(u^*) \cdot \mathbf{n}_i \tag{4}$$

where u^* is a suitable arbitrary average of u^h over the element, $\boldsymbol{\lambda}$ is the Jacobian of the flux $\mathscr{F}$ and $\mathbf{n}_i$ is the inward normal of the face opposite to the node i and scaled by the length of this face. Then a scheme is said to be upwind when

$$\Phi_i^{E_s} = 0 \quad \text{if } k_i < 0 \tag{5}$$

which means that the information is following the advection direction. The difficulty when considering a mesh with quadrilateral is that the inward normal is not defined intrinsically; indeed, there are two faces opposite to each node. We choose the normal which is the sum of the two normals of these faces as shown in Figure 1. Then, the upwind parameter is computed using Equation (4), and the definition of upwind schemes is the same (5). Moreover, since $\sum_{ij \in E} \mathbf{n}_{ij} = \mathbf{0}$ the residual is distributed to one (one target quadrilateral) or two nodes (two target): for more details see [1, 2].

The upwind parameter is defined in the same way when considering Q^2 elements; we just use now the normals of the sub-elements.

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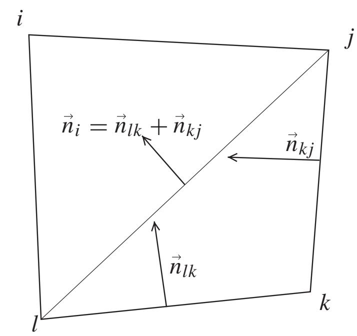

Figure 1. Inward normals to define the upwind parameter.

2. LINEAR SCHEMES

2.1. LDA Scheme

The general formula of the distribution coefficients of this scheme is

$$\beta_i^{E_s} = \frac{\max(0, k_i)}{\sum_{j \in E_s} \max(0, k_j)} \tag{6}$$

From this formula we note that this scheme is upwind. It is $(k+1)$-th accurate (on Q^k elements). Unfortunately, this scheme is not monotone.

2.2. N scheme

For this scheme the distributed residual is directly computed by

$$\Phi_i^{E_s} = \max(k_i, 0)(u_i - u_{\text{in}}) \tag{7}$$

where u_{in} is defined by

$$u_{\text{in}} = \frac{\sum_{i \in E_s} \min(0, k_i) u_i}{\sum_{j \in E_s} \min(0, k_j)} \tag{8}$$

This scheme using formula (8) is consistent only when using Q^1 elements on a parallelogram. When using non-Cartesian or high-order elements, this scheme is no longer consistent. Then a more general formula is used for u_{in}:

$$u_{\text{in}} = \frac{\Phi^{E_s} - \sum_{i \in E_s} \max(0, k_i) u_i}{\sum_{j \in E_s} \min(0, k_j)} \tag{9}$$

This scheme is monotone when using formula (8). When the more general one (9) is used, the scheme is no longer rigorously monotone, but it still behaves as a monotone scheme (see [3]). Unfortunately, it is always 1st order accurate (see the Godunov theorem [4]). We are interested in combining the accuracy of the LDA scheme and the monotonicity of the N scheme. A solution is

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to do blending between the LDA scheme and the N scheme in order to combine their advantages. Because of the Godunov theorem, this should work for non-linear schemes.

3. NON-LINEAR SCHEMES

3.1. Blending scheme

This scheme is based on the fact that we wish to have an accurate scheme on the smooth part of the solution and a monotone one on the discontinuous part. Then, the idea is to blend a monotone scheme (such as the N scheme) and an accurate one (such as the LDA). The scheme can be expressed as

$$\Phi_i^{E_s} = \theta\Phi_i^{\mathrm{N}} + (1-\theta)\Phi_i^{\mathrm{LDA}} \tag{10}$$

where θ is defined by

$$\theta = \frac{|\Phi^{E_s}|}{\sum_{i\in E_s}|\Phi_i^{E_s}|} \tag{11}$$

This scheme is not rigorously monotone, but it behaves almost as if it were. It is $(k+1)$-th accurate.

4. STABILIZATION

Unfortunately, these schemes show some instabilities on some of the vertices (see [1, 5]). This is due to poor convergence and also due to some oscillations on the solution. To cure these instabilities, an artificial dissipation is added. This stabilization is the extension of the one of Abgrall to high-order discretization:

$$\Phi_i^{\mathrm{stab}} = \int_E (\nabla\mathscr{F}(u_h)\cdot\nabla\psi_i)(\nabla\mathscr{F}(u_h)\cdot\nabla u^h)\,\mathrm{d}\Omega \tag{12}$$

where ψ_i is the biquadratic basis function of the node i. However, monotone schemes give good results around discontinuities without the stabilization; hence, we use a switch parameter to cancel the stabilization around discontinuities. Hence, the stabilization is multiplied by a coefficient α:

$$\alpha = 1 - \frac{|\max_{j\in E}(u_j) - \min_{j\in E}(u_j)|}{|\max_{j\in E}(u_j)| + |\min_{j\in E}(u_j)| + \varepsilon} \tag{13}$$

The stabilized LDA, N and B schemes are denoted by LDAs, Ns and Bs.

5. RESULTS

First, the accuracy of these schemes are shown in a practical problem. We take $\lambda = (1, \frac{1}{2})$ and the following boundary conditions:

$$u(0,y) = -\sin(2\pi y), \quad u(x,0) = \sin(\pi x)$$

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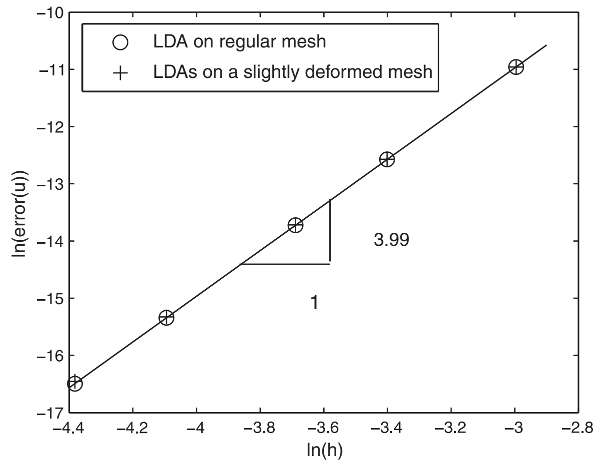

Figure 2. Grid convergence of LDA and LDAs on regular meshes.

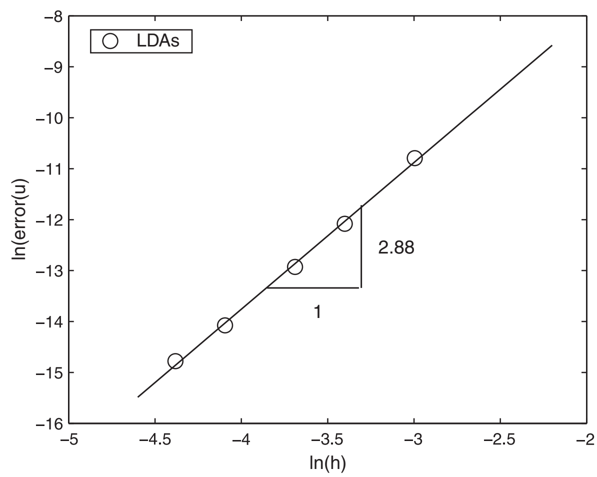

Figure 3. Grid convergence of LDAs on irregular meshes.

The exact solution to this problem is

$$\sin(\pi(x-2y))$$

We solve the problem with Q^2 elements on regular, slightly perturbed and unregular grids. Moreover, we compare the accuracy when using and not using the stabilization. We can see in Figures 2 and 3 that the LDA scheme is 4th order accurate on regular meshes, but it decreases to 3rd order on irregular ones. This may be explained by a term of the truncation error that cancels on Cartesian meshes. Moreover, we can see that the stabilization is not decreasing the accuracy. Then, in

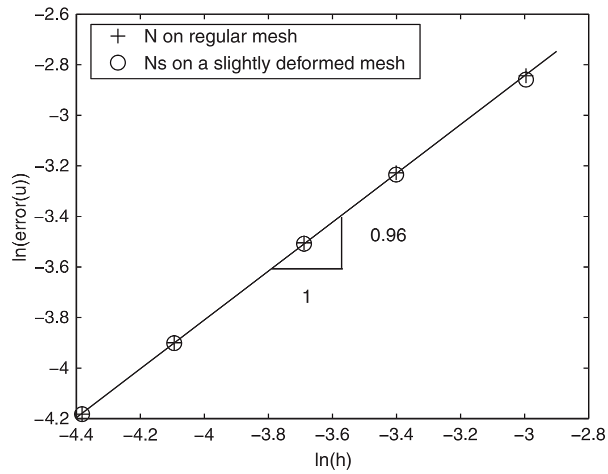

Figure 4. Grid convergence of N and Ns on regular meshes.

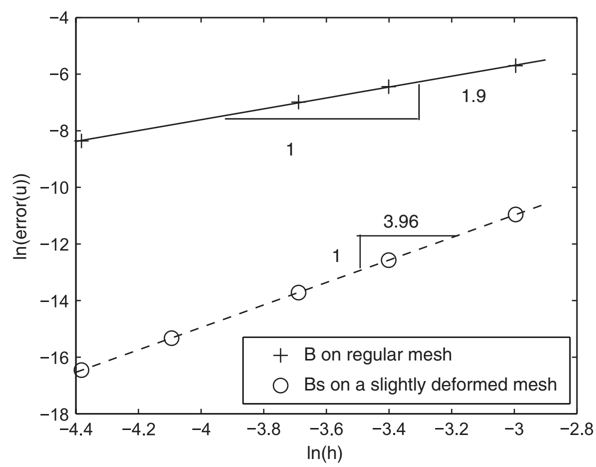

Figure 5. Grid convergence of B and Bs on regular meshes.

Figure 4 we can see that N scheme is 1st order accurate as expected. Finally, in Figures 5 and 6 we can see that, thanks to the blending the monotone scheme and the high-order scheme, the B scheme is 4th order accurate on regular meshes and 3rd on irregular ones. We can also note that the stabilization is really improving the accuracy. The goal of the second test case is to compare $\mathscr{RD}$ on Q^2 elements with the one on Q^1. This is done on the previous test case where the input is now

$$u(0, y) = -\sin(8\pi y), \quad u(x, 0) = \sin(4\pi x)$$

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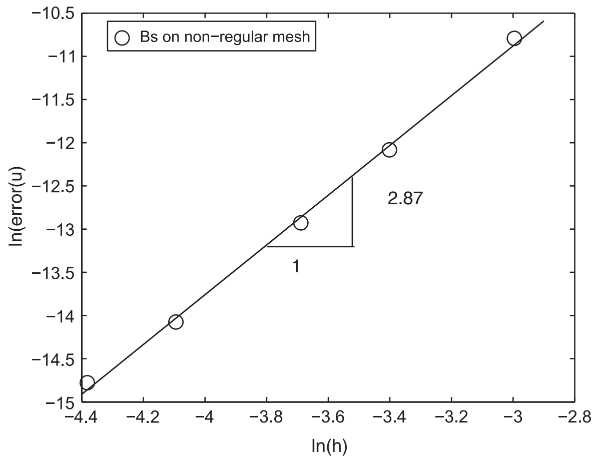

Figure 6. Grid convergence of Bs on irregular meshes.

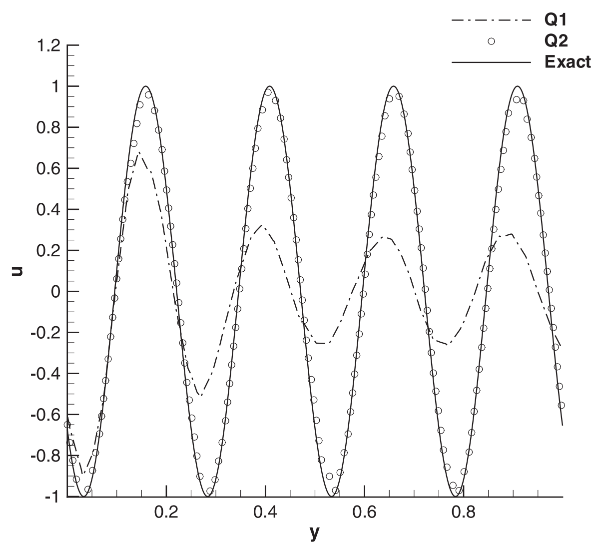

Figure 7. Comparison between schemes on Q^1 and Q^2 elements.

In Figure 7 we plot the cross section of the solution at $x=0.9$ computed by the Q^2 and Q^1 for the same degree of freedom. We can see that LDA on Q^2 elements has a better resolution on higher frequency than the one on Q^1 elements. Finally, the last test case is done on the Burger's equation: ($\mathscr{F}=(u^2/2,u)$) on the square $[0,1]^2$ with the boundary condition $u=1.5-2x$ for $y=0$. In Figure 8 we can see the contour levels obtained with the Bs scheme. Even if there are few oscillations, in Figure 9 we can see that the solution is improved in comparison with the LDA scheme.

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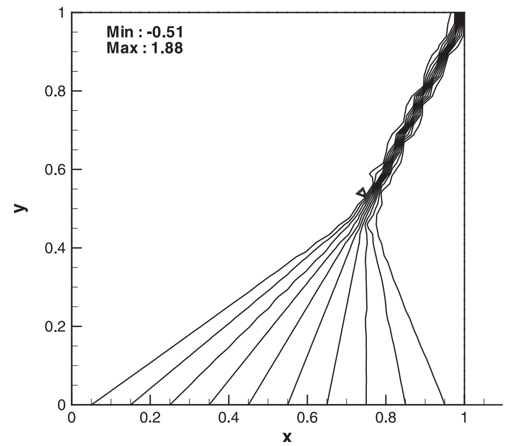

Figure 8. Line contour with Bs scheme.

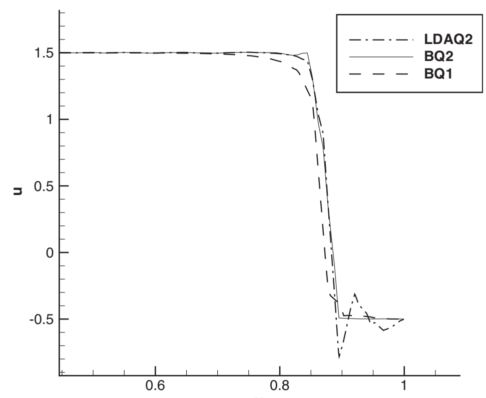

Figure 9. Comparison between LDAs and Bs.

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